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CREEPING FLOW OF AN ELLIS FLUID PAST A NEWTONIAN FLUID SPHERE

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Abstract—Upper and lower bounds on the drag offered to a Newtonian fluid sphere placed in an Ellis model fluid in creeping flow have been found using variational principles. For a solid sphere, the bounds are in good agreement with those reported by earlier investigators.

INTRODUCTION

The hydrodynamics of the fall of fluid spheres in quiescent non-Newtonian media has been studied by many investigators as this situation is encountered in many practical instances, such as in the production of penicilin and in treatment of sewage wastes. Simple rheological models, such as the power law model, are not nearly as interesting in representing real flow behavior because these fail to predict limiting viscosities. The Ellis model is a generally accepted three parameter model to describe the rheological behavior of most real fluids.

Variational principles for the flow of a generalized Newtonian fluid were first developed by Johnson (1961) and later by Slattery (1972) and Yoshioka & Adachi (1971). These principles have been used (Slattery 1961, 1962, 1972; Hopke & Slattery 1970; Mohan & Venkateswarlu 1974) to obtain approximate solutions of the drag offered to a solid sphere. Of special interest is the work of Hopke & Slattery (1970) who used the function space approach and presented both upper and lower bounds on the drag offered to a solid sphere placed in an Ellis fluid in creeping flow. A lower bound for the flow of an Ellis fluid past a fluid sphere was presented by Mohan & Venkateswarlu (1974). However, in this analysis the trace of the extra stress tensor was not set to zero.

In the present analysis, both upper and lower bounds on the drag offered to a Newtonian fluid sphere placed in an Ellis fluid in creeping flow have been found using variational principles.

ANALYSIS

The equations of continuity and motion can be written in tensorial notation as

$$\frac{\partial \rho}{\partial t} = -(\rho \mathbf{v}^i)_{,i}, \qquad [1]$$

$$\rho\left(\frac{\partial \mathbf{v}^i}{\partial t} + \mathbf{v}^j \mathbf{v}^i_{,j}\right) = -p^{,i} + \boldsymbol{\tau}^{ij}_{,j} + \rho \mathbf{f}^i, \qquad [2]$$

where ρ is the density, p the pressure, τ the extra stress tensor, v the velocity and f the body force.

The following assumptions are made:

1. The flow is steady, axisymmetric and creeping.
2. The fluid particle is spherical.
3. The fluid properties are constant.
4. The rheological behavior of the internal and external fluids are given by

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$$\boldsymbol{\tau} = 2\eta_i \mathbf{D} \qquad \text{(internal)},$$

$$\boldsymbol{\tau} = \frac{2}{\beta}\mathbf{D} \qquad \text{(external)}, \qquad [3a,b]$$

where

$$\beta = \frac{1}{\eta} = \frac{1}{\eta_0}\left\{1 + \left[\frac{\sqrt{(\Pi_\tau/2)}}{\tau_{1/2}}\right]^{\alpha-1}\right\}. \qquad [4]$$

D represents the rate-of-deformation tensor, η_i the viscosity of the internal Newtonian fluid, β the fluidity, η_0 the zero shear stress, α and $\tau_{1/2}$ the Ellis model parameters and Π_τ the second invariant of the extra stress tensor.

The flow being two dimensional and axisymmetric, a stream function ψ can be defined such that, in spherical polar coordinate system (R, θ, ϕ),

$$v_r = -\frac{1}{R^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad [5]$$

$$v_\theta = \frac{1}{R\sin\theta}\frac{\partial\psi}{\partial R}. \qquad [6]$$

This definition of the velocity components satisfies the equation of continuity automatically.

Using [3a], [5] and [6], the equation of motion for the internal fluid becomes

$$D^4\psi_i = 0, \qquad [7]$$

where

$$D^4 = \left\{\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right\}^2. \qquad [8]$$

The work function E, the complementary work function E_c and the functionals J_v and H_τ for the flow field are defined as (Slattery 1972)

$$E = \int_0^{\Pi} \eta \, \mathrm{d}\Pi, \qquad [9]$$

$$E_c = \int_0^{\Pi_\tau} \frac{\mathrm{d}\Pi_\tau}{4\eta}, \qquad [10]$$

$$J_v = \int_{V(s)} E^* \, \mathrm{d}V + \int_{(S-S_v)} (\mathbf{v} - \mathbf{v}^*) \cdot ([\mathbf{T} - \rho\phi\mathbf{I}] \cdot \mathbf{n}) \, \mathrm{d}S, \qquad [11]$$

$$H_\tau = -\int_{V(s)} E_c^* \, \mathrm{d}V + \int_S \mathbf{v} \cdot ([\mathbf{T} - \rho\phi\mathbf{I}]^* \cdot \mathbf{n}) \, \mathrm{d}S, \qquad [12]$$

where the quantities with superscript asterisk in [11] are obtained form trial velocity profiles that satisfy the equation of continuity and prescribed conditions on S_v where S_v is that part of the bounding surface S on which the velocity is explicitly specified. The quantities with a superscript asterisk in [12] are obtained from a trial stress profile that satisfies Cauchy's first law and prescribed boundary conditions for stress on S_t. $V(s)$ represents the flow domain. It was shown by Slattery (1972) that for single phase flows

$$J_v \geqslant \int_V \mathbf{E} \, \mathrm{d}V \geqslant H_\tau. \qquad [13]$$

The energy dissipation rate per unit volume is related to the work function E by

$$\mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v}) = 2E \qquad \text{(Newtonian fluids)}, \qquad [14]$$

$$2E \geq \mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v}) \geq \frac{\alpha+1}{\alpha}E \qquad \text{(Ellis fluids } \alpha \geq 1). \qquad [15]$$

Inequalities [13] to [15] combine to yield (for $\alpha \geq 1$),

$$2(J_v)_i \geq 2\int_{V_i} \mathbf{E}_i \,\mathrm{d}V = \int_{V_i} \mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v})_i \,\mathrm{d}V \geq 2(H_\tau)_i, \qquad [16]$$

$$2(J_v)_0 \geq 2\int_{V_0} \mathbf{E}_0 \,\mathrm{d}V \geq \int_{V_0} \mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v})_0 \,\mathrm{d}V \geq \frac{\alpha+1}{\alpha} \times \int_{V_0} \mathbf{E}_0 \,\mathrm{d}V \geq \frac{\alpha+1}{\alpha}(H_\tau)_0. \qquad [17]$$

Because of the use of inequality [15] in obtaining inequality [17], only a bound-on-bound is obtainable. Since for $\alpha \geq 1$, $(\alpha + 1/\alpha) \leq 2$, we have, on combining inequalities [16] and [17],

$$2\{(J_v)_i + (J_v)_0\} \geq \int_{V_i+V_0} \mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v})\,\mathrm{d}V \geq \frac{\alpha+1}{\alpha}\{(H_\tau)_i + (H_\tau)_0\}. \qquad [18]$$

Inequality [18] gives the bound-on-bound on the energy dissipation rate for the system as a whole.

Velocity principle

Trial stream functions are chosen in a form that reduce to the known solution for the Newtonian case:

$$\psi_i^* = (C_1 r^2 + C_3 r^4)(1 - Z^2)V_\infty a^2, \qquad [19]$$

$$\psi_0^* = \left(-\frac{1}{2}r^2 + A_1 r^\sigma + \frac{A_2}{r}\right)(1 - Z^2)V_\infty a^2. \qquad [20]$$

Equation [19] satisfies the differential equation [7] and the requirement that $(v_r)_i^*$ and $(v_\theta)_i^*$ remain finite as $r \to 0$. Equation [20] with σ equal to unity represents the flow of a Newtonian fluid past a Newtonian fluid sphere.

The boundary conditions on the flow are

$$\begin{aligned} (v_r)_i &= (v_r)_0 = 0 \quad \text{at} \quad r = 1, \\ (v_\theta)_i &= (v_\theta)_0 \qquad \text{at} \quad r = 1, \end{aligned} \qquad [21]$$

$$(\tau_{r\theta})_i = (\tau_{r\theta})_0 \qquad \text{at} \quad r = 1. \qquad [22]$$

Applying boundary conditions [21], equations [19] and [20] yield

$$\begin{aligned} C_1 + C_3 &= 0, \\ A_1 + A_2 &= 1/2, \\ \sigma A_1 - A_2 - 1 &= 2C_1 + 4C_3. \end{aligned} \qquad [23]$$

The boundary condition [22] is solved approximately using Galerkin's method (Finlayson 1972) which requires that

$$\int_{-1}^{1} (1-Z^2)\{(\tau_{r\theta})_i - (\tau_{r\theta})_0\}|_{r=1}^{*}\, \mathrm{d}Z = 0. \qquad [24]$$

For the trial profiles given by [19] and [20], [24] reduces to the form

$$\int_{-1}^{1} (1-Z^2)^{3/2}\left[\frac{(\sigma-1)(\sigma-2)A_1+6A_2}{\{1+(N_1S^{*1/2})_{x=1}^{\alpha-1}\}} - 6C_3X_E\right]\mathrm{d}Z = 0, \qquad [25]$$

where the viscosity ratio $X_E = \eta_i/\eta_0$ and S^* is the dimensionless second invariant of the stress tensor related to the dimensionless second invariant D^* of the rate-of-deformation tensor by

$$D^* = S^*\{1+(N_1S^{*1/2})^{\alpha-1}\}^2/4 \qquad [26a]$$

and for the trial stream function profile

$$D^* = x^4[6Z^2\{(2-\sigma)A_1x^{1-\sigma}+3A_2x^2\}^2$$
$$+(1-Z^2)\{(\sigma-2)(\sigma-1)A_1x^{1-\sigma}+6A_2x^2\}^2/2]. \qquad [26b]$$

From a macroscopic mechanical energy balance it can be shown that

$$V_\infty F_d = \int_{V_i+V_0} \mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v})\,\mathrm{d}V, \qquad [27]$$

where V_∞ is the free stream velocity and F_d the drag force. Equation [27] can be combined with inequality[18] and written in the dimensionless form

$$Y = \frac{C_dRe}{24} = \frac{1}{6\pi a\eta_0 V_\infty^2}\int_{V_i+V_0} \mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v})\,\mathrm{d}V$$
$$\leq \frac{(J_{v_i}+J_{v_0})}{3\pi a\eta_0 V_\infty}. \qquad [28]$$

The trial stream functions given by [19] and [20] have five arbitrary constants, four of which (A_1, A_2, C_1 and C_3) are related to σ by [23] and [25]. For any value of σ, the values of A_1, A_2, C_1 and C_3 are found using [23] and solving [25] by a Newton–Raphson iteration. The value of σ is chosen such that the R.H.S. of inequality [28] is a minium.

The internal fluid is bounded by the spherical interface $S_{i(r=1)}$ of radius a with the normal radially outward. The external fluid is bounded by the interface $S_{0(r=1)}$ with the normal radially inward, and a sphere $S_{0(r=\infty)}$ of radius infinity confining the unbounded fluid. The velocity is explicitly specified on this latter surface and therefore,

$$S_i = (S-S_v)_i = S_{i(r=1)},$$
$$S_0 = S_{0(r=1)} + S_{0(r=\infty)}, \qquad [29]$$
$$(S-S_v)_0 = S_{0(r=1)}$$

Since $v_r = v_r^* = 0$ at the interface, and since v_θ, v_θ^*, $\tau_{r\theta}$ and $\tau_{r\theta}^*$ are continuous across the interface, we have from [11] that

$$J_{v_i} + J_{v_0} = \int_{V_i} E_i^* \,\mathrm{d}V + \int_{V_0} E_0^* \,\mathrm{d}V. \quad [30]$$

Inequality [28] therefore becomes

$$Y = \frac{C_d R_e}{24} \leq \left\{ \int_{V_i} E_i^* \,\mathrm{d}V + \int_{V_0} E_0^* \,\mathrm{d}V \right\} \Big/ (3\pi a \eta_0 V_\infty^2). \quad [31]$$

Using the definition of the work function E given by [9], it can be shown that for the trial stream functions

$$Y \leq \frac{16}{3} C_1^2 X_E + \frac{1}{6}\int_{-1}^{1} \int_0^1 \left\{1 + \frac{2\alpha}{\alpha+1}(N_1 S^{*1/2})^{\alpha-1}\right\} \times S^* x^{-4} \,\mathrm{d}x \,\mathrm{d}x = F \quad [32]$$

where S^* is evaluated by solving [26]. The minimum of F is found by a Fibonacci search (Wilde 1964) on σ and this gives the upper bound Y_{UB}.

Stress principle

Trial extra stress tensors are chosen to be

$$\begin{aligned}
(\tau_{r\theta})_i^* &= 6C_3 r(1-Z^2)^{1/2}\left(\eta_i \frac{V_\infty}{a}\right),\\
(\tau_{\theta\theta})_i^* &= (\tau_{\phi\phi})_i^* = 4C_3 rZ\left(\eta_i \frac{V_\infty}{\alpha}\right),\\
(\tau_{rr})_i^* &= -2(\tau_{\theta\theta})_i^*.
\end{aligned} \quad [33]$$

$$\begin{aligned}
(\tau_{r\theta})_0^* &= -Ax^B(1-Z^2)^{1/2}\left(\eta_0 \frac{V_\infty}{a}\right),\\
(\tau_{rr})_0^* &= -(Cx^D + C'x^B)Z\left(\frac{\eta_0 V_\infty}{a}\right),\\
(\tau_{\theta\theta})_0^* &= -(Fx^D + F'x^B)Z\left(\frac{\eta_0 V_\infty}{a}\right),\\
(\tau_{\phi\phi})_0^* &= -(Ex^D + E'x^B)Z\left(\frac{\eta_0 V_\infty}{a}\right).
\end{aligned} \quad [34]$$

Since tr $[\boldsymbol{\tau}]$ is zero, [34] requires that

$$\begin{aligned}
C + F + E &= 0,\\
C' + F' + E' &= 0.
\end{aligned} \quad [35]$$

Further, by substituting the trial stress functions given by [34] into the equation of motion and equating

$$\frac{\partial^2}{\partial x \partial \theta}(p + \rho\phi)_0^* = \frac{\partial^2}{\partial \theta \partial x}(p + \rho\phi)_0^* \quad [36]$$

it can be shown that

$$\begin{aligned}
E &= F,\\
E' &= F',
\end{aligned}$$

$$D = 2,$$
$$C' = (B - 1)A + F'. \qquad [37]$$

The condition of jump momentum balance requires that the difference in the normal stress be related to the interfacial tension. This difference manifests itself as a pressure difference (Happel & Brenner 1965). However, the shear stress $\tau^*_{r\theta}$ is continuous at the interface and therefore from [33] and [34]

$$A = -6C_2X_E. \qquad [38]$$

Combining [28] and inequality [18] it can be shown that

$$Y = \frac{C_dRe}{24} = \frac{1}{6\pi a\eta_0 V_\infty^2}\int_{V_i + V_0} \mathrm{tr}\,(\boldsymbol{\tau}\cdot\nabla\mathbf{v})\,\mathrm{d}V$$
$$\geqslant \frac{\alpha + 1}{6\alpha}\{(H_\tau)_i + (H_\tau)_0\}\cdot(\pi a\eta_0 V_\infty^2). \qquad [39]$$

Since v_r is zero at the interface, and $\tau^*_{r\theta}$ and v_θ are continuous, [12] and [29] yield

$$(H_\tau)_i + (H_\tau)_0 = -\int_{V_i} E^*_{C_i}\,\mathrm{d}V - \int_{V_0} E^*_{C_0}\,\mathrm{d}V$$
$$+ \int_{S_{0(r=\infty)}} \mathbf{v}\cdot([\mathbf{T} - \rho\phi\mathbf{I}]^*\cdot\mathbf{n})\,\mathrm{d}S. \qquad [40]$$

Inequality [39] therefore becomes

$$Y = \frac{C_dRe}{24} \geqslant \frac{\alpha+1}{6\alpha}\left[-\frac{4}{3}(F - C) - 16C_1^2X_E - \frac{1}{2}\right.$$
$$\left.\times\int_{-1}^{1}\int_0^1\left\{1 + \frac{2}{\alpha+1}(N_1S^{*1/2})^{\alpha-1}\right\}S^*x^{-4}\,\mathrm{d}x\,\mathrm{d}Z\right] = H. \qquad [41]$$

The trial extra stress tensors given by [33] and [34] have ten arbitrary constants. Equations [35], [37] and [38] relate seven of these in terms of C_1, C and B, which are chosen by the method of Rosenbrock (Rosenbrock & Storey 1966) such that the quantity H in inequality [41] is a maximum. This gives the lower bound Y_{LB}.

RESULTS AND DISCUSSION

In figure 1 is plotted the variation of the upper and lower bounds with the viscosity ratio X_E and indicates that $10^{-2} \leqslant X_E \leqslant 10^3$ covers the range from the behavior of a bubble to that of a solid sphere.

Figure 2 presents a plot of the variation of the bounds with the Ellis parameter α for various values of X_E. The results of Hopke & Slattery (1970) for a solid sphere are shown for comparison with the results of the present investigation for the case of $X_E = 10^3$. The upper and lower bounds are seen to be quite close to those of Hopke & Slattery.

The variation of the bounds with α is shown in figure 3 for viscosity ratio $X_E = 1$ for various values of N_1. It is seen that for any N_1, the bounds diverge with increasing α. For $N_1 \to 0$, the upper bound tends to the Newtonian value of Y equal to $(2 + 3X_E)/(3 + 3X_E)$, while the lower bound approaches $(\alpha + 1)/2\alpha$ times the Newtonian value. For non-zero values of N_1, in the limit as $\alpha \to 1$, the bound approaches half the value of Y for a Newtonian fluid with $X = 2X_E$ since in this limit $\eta \to (\eta_0/2)$. This limit for Y is equal to $(1 + 3X_E)/(3 + 6X_E)$ and is indicated in figure 3 for the case $X_E = 1$.

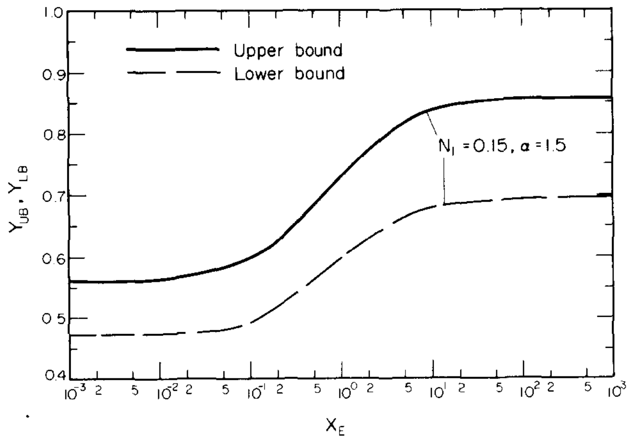

Figure 1. Variation of the bounds with the viscosity ratio.

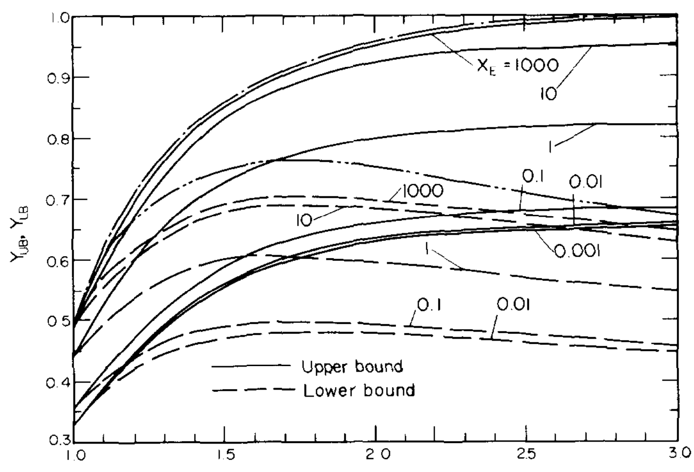

Figure 2. Variation of the bounds with the Ellis parameter α. Comparison with the results of Hopke & Slattery (1970) for a solid sphere — · — · — · — · upper bound, and — · · — · · — · · — · · lower bound.

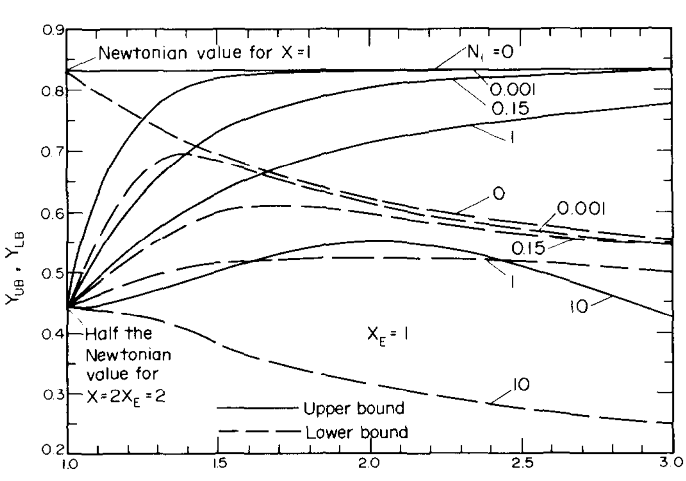

Figure 3. Effect of N_1 and α on the bounds.

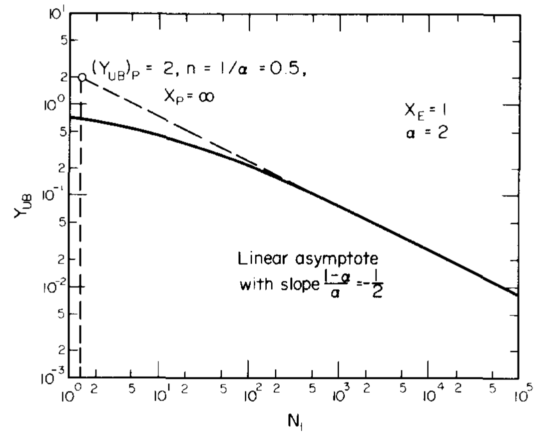

Figure 4. Asymptotic behavior of the upper bound at large N_1 for $\alpha = 2$.

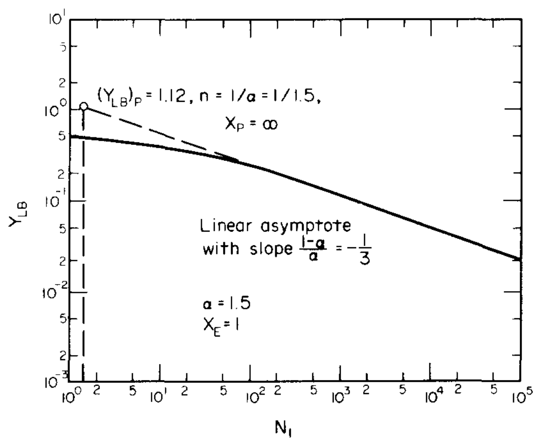

Figure 5. Asymptotic behavior of the lower bound at large N_1 for $\alpha = 1.5$.

It can be shown that for large values of N_1

$$Y \longrightarrow Y_p[N_1/\sqrt{(2)}]^{(1-\alpha)/\alpha} \qquad [42]$$

where $Y_p = C_d \cdot Re_p/24$ is the value of Y for the flow of a power law fluid ($n = 1/\alpha$) past a solid sphere. This asymptotic behavior of the bound at large N_1 is shown in figures 4 and 5 for the upper bound with $\alpha = 2$ and $X_E = 1$ and for the lower bound with $\alpha = 1.5$ *and* $X_E = 1$.

CONCLUSIONS

(1) The bounds on the drag are close for values of X_E near unity and diverge at higher values of α for all values of N_1.

(2) Asymptotic behavior of the bounds indicate that Newtonian values are obtained as $N_1 \to 0$ or $\alpha \to 1$ and that the results for the flow of a power law fluid past a solid sphere are predicted in the limit $N_1 \to \infty$.

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REFERENCES

FINLAYSON, B. A. 1972 *The Method of Weighted Residuals and Variational Principles with Application in Fluid Mechanics, Heat and Mass Transfer*, pp. 278–285. Academic Press, New York.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*, p. 127. Prentice Hall, New Jersey.

HOPKE, S. W. & SLATTERY, J. C. 1970 Upper and lower bounds on the drag coefficient of a sphere in an Ellis model fluid. *A.I.Ch.E. Jl* **16**, 224–229.

JOHNSON, M. W., JR. 1961 On variational principles for non-Newtonian fluids. *Trans. Soc. Rheol.* **5**, 9–21.

MOHAN, V. & VENKATESWARLU, D. 1974 Lower bound on the drag offered to a Newtonian fluid sphere placed in a flowing Ellis fluid, *J. Chem. Engng, Jap.* **7**, 243–247.

ROSENBROCK, H. H. & STOREY, C. 1966 *Computational Methods for Chemical Engineers*, p. 64. Pergamon Press, Oxford.

SLATTERY, J. C. 1961 Flow of a simple non-Newtonian fluid past a sphere. *Appl. Scient.* Res. **A10**, 286–296.

SLATTERY, J. C. 1962 Approximations to the drag force on a sphere moving slowly through either an Ostwald-de-Waele or a Sisko fluid. *A.I.Ch.E. Jl* **8**, 663–667.

SLATTERY, J. C. 1972 *Momentum, Energy and Mass Transfer in Continua*, p. 260. McGraw-Hill, New York.

WILDE, D. J. 1964 *Optimum Seeking Methods*, p. 10. Prentice Hall, New York.

YOSHIOKA, N. & ADACHI, K. 1971 On variational principles for a non-Newtonian fluid, *J. Chem. Engng, Jap.* **4**, 217–220.

YOSHIOKA, N. & ADACHI, K. 1971 Applications of the extremum principles for non-Newtonian fluids. *J. Chem. Engng, Jap.* **4**, 221–226.